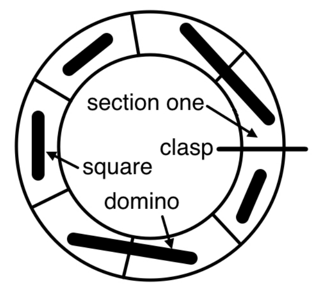
Team Play Topics

Round Two

The first section of the Round Two Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: The *Lucas numbers* are $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, ... in which each term of the sequence from L_3 on is equal to the sum of the previous two terms. It is known that L_n counts the number of ways to *tile a bracelet with clasp* of length n with squares and dominoes. A bracelet of length 7 (having seven sections) is shown along with the tiling $dssds$, where d represents 'domino,' s stands for 'square,' and we list the tiles in counterclockwise order, beginning with the tile covering section one. We obtain the new tiling $sdssd$ by moving all tiles one section counterclockwise. Rotating the tiles clockwise instead gives the new tiling $Dssds$, where we capitalize the first D to indicate that the first domino now covers the clasp.



Topics: Lucas numbers, combinatorial proof

Practice Problems

1. Compute Lucas numbers L_5 through L_{10}. How should we define L_0 to be consistent with our method of computing each Lucas number from the previous two?

2. Confirm that there are indeed L_4 tilings of a bracelet with clasp of length 4. Do so by listing all seven such tilings using the notation described above.

3. How many tilings of a bracelet of length 7 should there be? (Three such tilings are mentioned above.) Convince yourself that your answer is correct by listing all such tilings.

4. How many tilings of a bracelet of length n have a square as the last tile? How many have a domino as the last tile? Why does this make sense?

5. Let s_n, d_n and D_n count the number of tilings of a bracelet of length n that start with an s, d and D, respectively. Explain why $s_n + 2d_n = L_n$.

6. Continuing the previous problem, show that $d_n = s_{n-1}$. What else can you discover about the numbers s_n and d_n?

Hints and answers on the next page. $\Longrightarrow$

Team Play Topics

HINTS AND ANSWERS

1. The first ten Lucas numbers are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123. We should set $L_0 = 2$, so that $L_0 + L_1 = L_2$ works as $2 + 1 = 3$.

2. The seven tilings are *ssss*, *dss*, *Dss*, *sds*, *ssd*, *dd* and *Dd*. It may be helpful to draw sketches of these tilings—be sure to understand the difference between *dss* and *Dss*, for instance. (Note that the only d that can ever be capitalized is the first one.)

3. There should be $L_7 = 29$ tilings. We organize our list according to the number of dominoes used: either zero, one, two, or three dominoes.

(0) *sssssss*

(1) *dsssss*, *Dsssss*, *sdssss*, *ssdsss*, *sssdss*, *ssssds*, *sssssd*

(2) *ddsss*, *Ddsss*, *dsdss*, *Dsdss*, ... (ten tilings with no D, plus four beginning with D)

(3) *ddds*, *Ddds*, *ddsd*, *Ddsd*, *dsdd*, *Dsdd*, *sddd*

4. By removing the section of the bracelet occupied by the last square, we obtain a tiling of a bracelet of length $n-1$. (Note that this works even when the clasp is covered by the first domino.) Hence there are L_{n-1} tilings that end with a square. Similarly, removing a final domino yields a tilling of a bracelet of length $n-2$, for a total of L_{n-2} ways. This gives all possible tilings, of which there are L_n. Hence $L_n = L_{n-1} + L_{n-2}$, which agrees with the recurrence for the Lucas numbers.

5. The tilings counted by s_n, d_n and D_n among them include all tilings of a bracelet of length n, hence $s_n + d_n + D_n = L_n$. But clearly $d_n = D_n$, since we can always move a tiling starting with d one section clockwise to obtain one starting with a D, giving $s_n + 2d_n = L_n$.

6. Given a tiling counted by d_n, we replace the initial domino by a square, giving a tiling of a bracelet of length $n - 1$ instead. In this way we find that $d_n = s_{n-1}$. There are many nice relationships to be found, including $s_{n+1} = s_n + d_n$, (see why?) which leads to $s_{n+1} = s_n + s_{n-1}$ using the result of this problem. In fact, the numbers s_n are the Fibonacci numbers, as are the values of d_n, ultimately leading to the identity $F_n + 2F_{n-1} = L_n$.