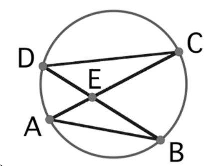
Team Play Topics

ROUND THREE

The first section of the Round Three Mandelbrot Team Play is reproduced below. A list of topics and practice problems are also provided to aid in preparation. Note that these problems are not meant to serve as a precise indicator of the problems that will appear on the contest. However, students who understand how to solve them should be able to make significantly more progress than they might have otherwise. So work hard on the problems, and good luck on the Team Play!

Facts: The *Intercepted Arc Theorem* states that if points A, B and C are on a circle, then $m\angle CAB = \frac{1}{2}m\widehat{BC}$. In other words, an angle inscribed in a circle measures half its intercepted arc. An extension of this result applies to angles within a circle; in the diagram at right we have $m\angle BEC = \frac{1}{2}(m\widehat{AD} + m\widehat{BC})$.



Let W be a point on a given segment $\overline{UV}$ other than its midpoint. Then the set of points Z in the plane such that $ZU/ZV = WU/WV$ is a *circle of Apollonius*, whose center lies on line UV. (Only needed for the final part.)

TOPICS: angles and intercepted arcs, similar triangles, the incenter, Angle Bisector Theorem, orthogonal circles, circle of Apollonius

Practice Problems

1. Let points A, B and C lie on a circle, and suppose the angle bisector of $\angle ABC$ crosses the circle at point M. Explain why M is the midpoint of arc $\widehat{AC}$.

2. In the previous part, let the perpendicular from A to $\overline{BC}$ intersect the circle again at X, and let the perpendicular from B to $\overline{AC}$ intersect the circle again at Y. Show that $m\widehat{CX} = m\widehat{CY}$.

3. Now draw $\overline{XY}$, which meets $\overline{BC}$ at D and meets $\overline{AC}$ at E. Prove that $\angle CDE \cong \angle BAC$.

4. Finally, in the diagram from the previous part, find a pair of similar triangles and use them to prove that $(CA)(CE) = (CB)(CD)$.

5. Given a point P outside a circle ω, explain how to construct with straight-edge and compass a circle with center at P which is orthogonal to ω.

Hints and answers on the next page. $\Longrightarrow$

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Hints and Answers

1. By the Intercepted Arc Theorem we know that $m\angle ABM = \frac{1}{2}m\widehat{AM}$ and $m\angle MBC = \frac{1}{2}m\widehat{MC}$. But since $\angle ABM \cong \angle MBC$ due to the angle bisector, this implies that $m\widehat{AM} = m\widehat{MC}$, meaning that M is the midpoint.

2. First observe that $\angle CAX \cong \angle CBY$, since these angles are both complementary to $\angle ACB$. Now use intercepted arcs to see that $m\widehat{CX} = m\widehat{CY}$.

3. By intercepted arcs, we know that $m\angle BAC = \frac{1}{2}(m\widehat{BX} + \widehat{XC})$. On the other hand, by the Facts section we have $m\angle CDE = \frac{1}{2}(m\widehat{BX} + m\widehat{CY})$. Now use the previous result to finish.

4. It is clear by AA similarity that $\triangle ABC \sim \triangle DEC$, using the pair of congruent angles above along with the common angle shared at C. Thus corresponding sides have the same ratio, meaning $\frac{CA}{CB} = \frac{CD}{CE}$. Cross-multiplying gives the desired equality.

5. The circles will be orthogonal as long as they cross at right angles, meaning that the radii to the point of intersection are perpendicular. Therefore let O be the center of ω, draw segment $\overline{PO}$, plot its midpoint M, then draw the circle with center M and radius MP. This is *not* the circle we want, but if it intersects ω at Q, then the circle with center P and radius PQ is the desired circle, since the construction guarantees that $\overline{PQ} \perp \overline{OQ}$.