January 2009

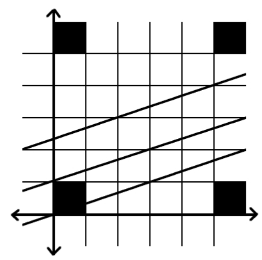
The Mandelbrot Team Play

Round One Test

Time Limit: 60 minutes

Setup: consider an infinite sheet of graph paper in which every fifth square is filled in. More precisely, plot every point (x, y) in the Cartesian plane with $5c \leq x \leq 5c + 1$ and $5d \leq y \leq 5d + 1$ for integers c and d, meaning that the x and y coordinates are each at most one more than a multiple of five. We call this collection of squares a *lattice*, and refer to a given square by the coordinates of its lower left corner. A portion of the lattice is shown below, including the $(0, 0)$-square, the $(5, 0)$-square, the $(0, 5)$-square and the $(5, 5)$-square.

Now let $y = mx + b$ be the equation of a line. We say that this line *cleanly threads* the lattice if it does not intersect any of the squares in the lattice at all. Next, we say that the line *barely threads* the lattice if the line contains the vertices of some of the squares, but does not pass through their interiors. Finally, if a line crosses into the interior of any square then it does not thread the lattice. These three situations are illustrated by the top, middle, and bottom lines in the diagram.



Problems

Part i: (4 points) Find a line with slope 1 that threads the lattice. Give the equation of such a line and sketch it in the plane along with at least four nearby lattice squares. Now do the same for lines having slopes $-\frac{1}{2}$, $\frac{2}{3}$ and 3. Which line only barely threads the lattice?

Part ii: (5 points) Show that if there is a line with slope m that cleanly threads the lattice, then there are lines with slope $-m$, $\frac{1}{m}$ and $-\frac{1}{m}$ that also cleanly thread the lattice. (You do not need to work with equations of lines here; a geometric argument is sufficient.)

Part iii: (4 points) Suppose that light rays with slope $\frac{3}{4}$ shine from the right. Describe the shadow cast onto the y-axis by the $(5, 5)$-square. (Give its endpoints and length.)

Part iv: (5 points) Continuing the previous part, demonstrate that there is another square in the lattice whose shadow is slightly higher than but overlaps with that of the $(5, 5)$-square. Extend these ideas to prove that a slope $\frac{3}{4}$ line cannot thread the lattice.

Part v: (5 points) In general, let $m = \frac{p}{q}$ be a slope which is a positive rational number, and consider light rays with slope m. Prove that if $p + q > 5$ then there exists a square in the lattice whose shadow on the y-axis overlaps with the shadow cast by the $(0, 0)$-square.

Part vi: (5 points) Let $m = \frac{p}{q}$ be a positive fraction with $p + q > 5$ as before. Prove that a slope m line cannot thread the lattice. Finally, conjecture what happens if $p + q = 5$.