February 2009

The Mandelbrot Team Play

Round Two Test

Time Limit: 60 minutes

Facts: The Fibonacci numbers are the sequence of integers 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ... in which each number is the sum of the two previous numbers. The k^{th} Fibonacci number is written F_k, so this sequence satisfies $F_0 = 0$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$ for $k \geq 1$.

Setup: We will write "k Fibonacci-factorial," denoted as $FF(k)$, to represent the product of the k Fibonacci numbers from F_k down to F_1. Thus $FF(5) = 5 \cdot 3 \cdot 2 \cdot 1 \cdot 1 = 30$. We also declare that $FF(0) = 1$. We can now define $\left(\!\binom{n}{k}\!\right)$, called a *Fibonomial*, as

$$\left(\!\!\binom{n}{k}\!\!\right) = \frac{FF(n)}{FF(k) \cdot FF(n-k)}.$$

Thus $\left(\!\binom{5}{3}\!\right) = \frac{FF(5)}{FF(3) \cdot FF(2)} = \frac{30}{2 \cdot 1} = 15$. One can also confirm that $\left(\!\binom{7}{6}\!\right) = 13$ and that $\left(\!\binom{0}{0}\!\right) = 1$.

Problems

Part i: (4 points) One of the more delightful relationships among the Fibonacci numbers states that $F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1}$. To begin, explain why this statement is true when $b = 0$ or $b = 1$. Also confirm the equality when $a = 5$ and $b = 7$.

Part ii: (5 points) Now prove that $F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1}$ holds for all $a \geq 0$ and all $b \geq 0$ by using induction on b. (The previous part demonstrates the base cases.)

Part iii: (5 points) By choosing a strategic value for b in the above relationship, show that for $a \geq 1$, F_{2a} is a multiple of F_a. Based on this, next show that F_{3a} is also a multiple of F_a.

Part iv: (4 points) There is a "Fibonomial triangle" whose n^{th} row consists of the numbers $\left(\!\binom{n}{0}\!\right)$, $\left(\!\binom{n}{1}\!\right)$, up to $\left(\!\binom{n}{n}\!\right)$. Compute rows $n = 0$ through $n = 7$, listing them on top of one another in a triangular arrangement. (For example, row $n = 3$ has the numbers 1 2 2 1.)

Part v: (5 points) Each row of the Fibonomial triangle can be used to generate the next one via the identity $\left(\!\binom{n+1}{k+1}\!\right) = F_{n-k+1} \left(\!\binom{n}{k}\!\right) + F_k \left(\!\binom{n}{k+1}\!\right)$. Prove this identity for all $n > k \geq 0$. Why does this imply that Fibonomials are always integers?

Part vi: (5 points) Prove that $F_{n+1}^2 = 2F_n^2 + 2F_{n-1}^2 - F_{n-2}^2$ for all $n \geq 2$. It is also true that $F_{n+1}^3 = 3F_n^3 + 6F_{n-1}^3 - 3F_{n-2}^3 - F_{n-3}^3$, but you do *not* need to prove this fact. Based on these examples, conjecture a formula for F_{n+1}^4. Test your conjecture when $n = 5$.