March 2009

# The Mandelbrot Team Play

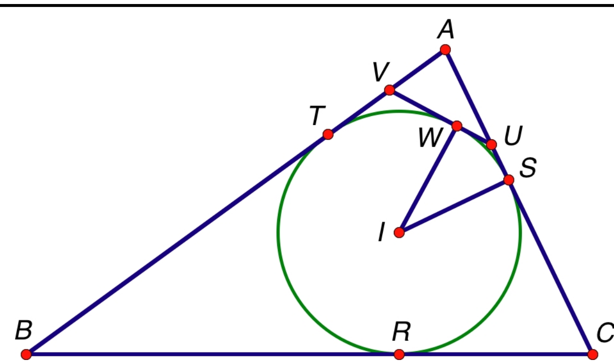
## Round Three Test

*Time Limit: 60 minutes*

**Facts:** For an arbitrary triangle $\triangle ABC$ with side lengths $BC = a$, $AC = b$ and $AB = c$ let $s = \frac{1}{2}(a+b+c)$ be the semiperimeter and define $s_a = s - a$, $s_b = s - b$ and $s_c = s - c$. Let $r$ be the radius of the circle inscribed in $\triangle ABC$. Also let $r_a$ be the radius of the excircle which is located outside $\triangle ABC$ tangent to both sides of angle $\angle BAC$ and to side $\overline{BC}$. We define $r_b$ and $r_c$ analogously. Finally, let $K$ be the area of $\triangle ABC$.

There are a remarkable number of relationships among these nine quantities, including $K = rs = r_a s_a = r_b s_b = r_c s_c$, $K = \sqrt{s s_a s_b s_c}$, $r r_a = s_b s_c$, $r r_b = s_a s_c$ and $r r_c = s_a s_b$. You may use any of these equalities in your solutions to the problems below.

**Setup:** In $\triangle ABC$ draw the inscribed circle with center $I$, tangent to the sides at points $R$, $S$ and $T$ as shown. Then draw segment $\overline{UV}$ tangent to the incircle at $W$ with $U$ on $\overline{AC}$ and $V$ on $\overline{AB}$ so that $\angle AUV \cong \angle ABC$. As demonstrated in the practice problems, we know that $AS = AT = s_a$, $BR = BT = s_b$, $CR = CS = s_c$, $US = UW$, and $VT = VW$.




**Problems**

**Part i: (4 points)** Show that $\triangle ISU \cong \triangle IWU$, then explain why $m\angle SIU = \frac{1}{2} m\angle B$.

**Part ii: (4 points)** Prove that $\triangle ISU \sim \triangle BRI$ and use this to deduce that $SU = r^2/s_b$.

**Part iii: (5 points)** Find a similar expression for $TV$, then show that $UV = ar/r_a$.

**Part iv: (5 points)** Prove that $\dfrac{b}{c} = \dfrac{s_a - (r^2/s_c)}{s_a - (r^2/s_b)}$. (Tip: try a geometric approach.)

**Part v: (5 points)** Demonstrate that $\mathit{area}(BCUV) = (UV)(r + r_a)$.

**Part vi: (5 points)** Finally, establish that $\mathit{area}(BCUV) = \sqrt{(BC)(BV)(CU)(UV)}$. (You should base your explanation on the above results. Please do not use a fancy area formula unless you include its proof along with your solution.)