Answer Key

1. $110°$
2. 8
3. $6 - 2i$
4. $\frac{2}{3}$
5. $186\frac{2}{3}$
6. 1996
7. 42

1. The three intersecting lines form a triangle. We can easily compute its two lower angles, since they are supplementary to the given angles. We find their measures to be $180° - 169° = 11°$ and $180° - 121° = 59°$. But the angles of the triangle add to $180°$, so the upper angle has measure $180° - 11° - 59° = 110°$. Finally, the angle indicated by the question mark is a vertical angle to the upper angle, so it also has measure $\mathbf{110°}$.

2. Carolina may extend her right hand to shake either of Dakota's or Montana's hands, for a total of four options. Regardless of whom she chooses, she must then turn to the other person (since that person is not allowed to shake their own hand) and shake one of their hands, which can happen in two ways. Hence Carolina can select her handshakes in a total of $4 \cdot 2 = 8$ ways. The third handshake between Dakota and Montana can occur in only one way, so the overall answer is also **8**.

3. Since $\alpha + \beta$ is a real number, α must involve a $-2i$ to cancel the $2i$ from β. Thus α is of the form $\alpha = a - 2i$ for some real number a. We then find that

$$i(\alpha - 2\beta) = i(a - 2i - 6 - 4i) = 6 + i(a - 6).$$

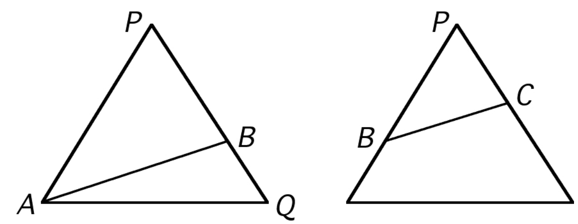
Since this is also a real number, we must choose $a = 6$ so that the imaginary parts will cancel. We deduce that $\alpha = \mathbf{6 - 2}\boldsymbol{i}$. (Note that we never needed to use the fact that we had *positive* real numbers.)

4. We begin by listing all six possible preference orders:

$$PGY \quad PYG \quad GPY \quad GYP \quad YPG \quad YGP.$$

The first option means that Jordan likes purple jellybeans the most, then green, then yellow the least, and similarly for the others. Once we know that he prefers purple over yellow the three options to the right are eliminated, leaving only the three options on the left. In two of them he prefers green over yellow, so the desired probability is $\boldsymbol{\frac{2}{3}}$.

5. The front and right faces of the pyramid are pictured below. Thus the trail begins at A, slants upward to B, turns the corner, continues on to C, and so forth.



The statement of the problem indicates that $AQ = 80$, $QB = 30$, and $m\angle AQP = 60°$, since $\triangle PAQ$ is equilateral. By the Law of Cosines,

$$\begin{aligned}(AB)^2 &= 80^2 + 30^2 - 2(80)(30)\cos 60° \\ &= 7300 - 2400 = 4900,\end{aligned}$$

thus $AB = 70$. We next notice that triangles PAB and PBC are similar, since $\overline{AB}$ and $\overline{BC}$ slant upwards by the same amount and the angles at P are congruent. But $PA = 80$ while $PB = 50$, so the second triangle is $\frac{5}{8}$ the size of the first. In particular, it follows that $BC = \frac{5}{8}(AB)$. In fact, each segment of the trail will be $\frac{5}{8}$ the length of the previous one, by the same reasoning. Hence we can compute the total length of the trail using the sum of a geometric series:

$$70 + \tfrac{5}{8}(70) + \tfrac{5}{8}(\tfrac{5}{8}(70)) + \cdots = \frac{70}{1 - \frac{5}{8}} = \frac{560}{3} = \mathbf{186\tfrac{2}{3}}.$$

Alternately, one can simply "unwrap" the entire path into a plane and note that the total length of the path is exactly $8/3$ times as long as segment $\overline{AB}$. (Do you see why? Use similar triangles.) Hence the total length is again $\frac{8}{3}(70) = 186\frac{2}{3}$.

6. Without loss of generality we may assume that one of the vertices of the square is the origin $A(0,0)$, while the second is the point $B(m,n)$ in the first quadrant, so that m and n are positive integers. Since the square has area 2009 we know that $AB = \sqrt{2009}$, hence $m^2 + n^2 = 2009$ by the Pythagorean Theorem. But $2009 = 7^2 \cdot 41$, so that $7|(m^2+n^2)$. One can prove that this implies that $7|m$ and $7|n$. (The reader is invited to fill in the details. One approach involves checking all possible values of $m^2 + n^2$ as m and n range from 0 to 6 mod 7.) So we write $m = 7m'$ and $n = 7n'$, which leads to the equation $(m')^2 + (n')^2 = 41$. The only solutions are $m' = 4$, $n' = 5$ or vice-versa; without loss of generality we choose the former, giving $m = 28$, $n = 35$.

We now employ Pick's Theorem, which states that the area of the square is given by $I + \frac{1}{2}B - 1$, where I and B are the number of lattice points in the interior and along the boundary, respectively. It is not hard to confirm that there are six lattice points between A and B on $\overline{AB}$, and similarly for the other edges, leading to $B = 28$ (including the four vertices). Therefore $I + 14 - 1 = 2009$, or $I = \mathbf{1996}$.

7. Every now and again it is fun to pull solutions out of a hat, as we shall do here. Note that

$$(a-5)(b-5)(c-5) = abc - 5ab - 5ac - 5bc + 25(a+b+c) - 125.$$

But we know that $(a+b+c) = 79$ and that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{5} \qquad \Longrightarrow \qquad 5ab + 5ac + 5bc = abc.$$

Therefore the first four terms of the above expression cancel, and we are left with

$$(a-5)(b-5)(c-5) = 25(79) - 125 = 25(79-5) = 2 \cdot 5 \cdot 5 \cdot 37.$$

At this point there are very few possibilities for the values of $(a-5)$, $(b-5)$, and $(c-5)$, since they must divide the right-hand side. We quickly discover the unique solution $a = 7$, $b = 30$, and $c = \mathbf{42}$.

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⋆ National Level ⋆

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