Answer Key	
1. 26	4. $6 - 2i$
2. 110°	5. $\frac{2}{3}$
3. 8	6. $186\frac{2}{3}$
	7. $2\sqrt{7} - 1$

1. There are relatively few ways to obtain 160 wheels using a combination of 18-wheeled tractor trailers and 4-wheeled cars. Thus there might be 40 cars, but this results in 40 antennae, which is too many. However, we can always replace nine cars by two tractor trailers without changing the number of wheels present. This swap has the effect of decreasing the number of antennae by five. So performing two such exchanges gives the answer of 22 cars and 4 tractor trailers, for a total of **26** vehicles.

2. The three intersecting lines form a triangle. We can easily compute its two lower angles, since they are supplementary to the given angles. We find their measures to be $180^\circ - 169^\circ = 11^\circ$ and $180^\circ - 121^\circ = 59^\circ$. But the angles of the triangle add to 180°, so the upper angle has measure $180^\circ - 11^\circ - 59^\circ = 110^\circ$. Finally, the angle indicated by the question mark is a vertical angle to the upper angle, so it also has measure $\mathbf{110^\circ}$.

3. Carolina may extend her right hand to shake either of Dakota's or Montana's hands, for a total of four options. Regardless of whom she chooses, she must then turn to the other person (since that person is not allowed to shake their own hand) and shake one of their hands, which can happen in two ways. Hence Carolina can select her handshakes in a total of $4 \cdot 2 = 8$ ways. The third handshake between Dakota and Montana can occur in only one way, so the overall answer is also **8**.

4. Since $\alpha + \beta$ is a real number, α must involve a $-2i$ to cancel the $2i$ from β. Thus α is of the form $\alpha = a - 2i$ for some real number a. We then find that

$$i(\alpha - 2\beta) = i(a - 2i - 6 - 4i) = 6 + i(a - 6).$$

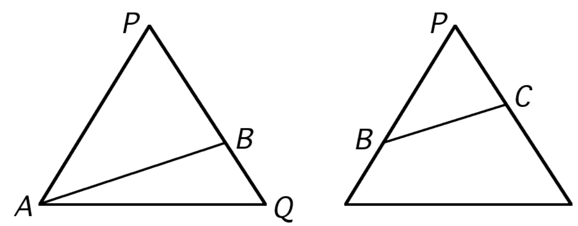
Since this is also a real number, we must choose $a = 6$ so that the imaginary parts will cancel. We deduce that $\alpha = \mathbf{6 - 2i}$. (Note that we never needed to use the fact that we had *positive* real numbers.)

5. We begin by listing all six possible preference orders:

$$PGY \quad PYG \quad GPY \quad GYP \quad YPG \quad YGP.$$

The first option means that Jordan likes purple jellybeans the most, then green, then yellow the least, and similarly for the others. Once we know that he prefers purple over yellow the three options to the right are eliminated, leaving only the three options on the left. In two of them he prefers green over yellow, so the desired probability is $\mathbf{\frac{2}{3}}$.

6. The front and right faces of the pyramid are pictured below. Thus the trail begins at A, slants upward to B, turns the corner, continues on to C, and so forth.



The statement of the problem indicates that $AQ = 80$, $QB = 30$, and $m\angle AQP = 60^\circ$, since $\triangle PAQ$ is equilateral. By the Law of Cosines,

$$\begin{aligned}(AB)^2 &= 80^2 + 30^2 - 2(80)(30)\cos 60^\circ \\ &= 7300 - 2400 = 4900,\end{aligned}$$

thus $AB = 70$. We next notice that triangles PAB and PBC are similar, since $\overline{AB}$ and $\overline{BC}$ slant upwards by the same amount and the angles at P are congruent. But $PA = 80$ while $PB = 50$, so the second triangle is $\frac{5}{8}$ the size of the first. In particular, it follows that $BC = \frac{5}{8}(AB)$. In fact, each segment of the trail will be $\frac{5}{8}$ the length of the previous one, by the same reasoning. Hence we can compute the total length of the trail using the sum of a geometric series:

$$70 + \tfrac{5}{8}(70) + \tfrac{5}{8}(\tfrac{5}{8}(70)) + \cdots = \frac{70}{1-\frac{5}{8}} = \frac{560}{3} = \mathbf{186\tfrac{2}{3}}.$$

Alternately, one can simply “unwrap” the entire path into a plane and note that the total length of the path is exactly 8/3 times as long as segment $\overline{AB}$. (Do you see why? Use similar triangles.) Hence the total length is again $\frac{8}{3}(70) = 186\frac{2}{3}$.

7. For sake of convenience, let us denote the values of the first two terms by a and b. (We are given that $a = 1+\sqrt{7}$, but we will not use this fact yet.) Using the relation $a_n = a_{n-1}a_{n+1}$ we may compute subsequent values in terms of a and b, yielding the sequence

$$a,\ b,\ \frac{b}{a},\ \frac{1}{a},\ \frac{1}{b},\ \frac{a}{b},\ a,\ b, \ldots$$

Clearly the sequence begins to repeat at this point. In other words, every sixth term is equal to $\frac{a}{b}$, including a_{1776}. Thus $a = 1+\sqrt{7}$ while $\frac{a}{b} = 13+\sqrt{7}$. Since 2009 is 5 more than a multiple of 6, we find that

$$a_{2009} = \frac{1}{b} = \frac{a}{b}\cdot\frac{1}{a} = \frac{13+\sqrt{7}}{1+\sqrt{7}} = \frac{(13+\sqrt{7})(\sqrt{7}-1)}{6} = \mathbf{2\sqrt{7}-1}.$$

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