ANSWER KEY

1. $\frac{10}{11}$
2. A
3. 7
4. -3
5. 6
6. $\frac{10}{3}$
7. 15

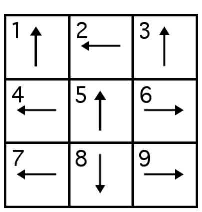
1. We can regroup and rewrite the sum on top as

$$(1+19)+(2+18)+\cdots+10=(10+10)+(10+10)+\cdots+10.$$

Since there were 19 terms in the top sum, its value is $19\cdot 10$. By the same reasoning, the bottom sum is $19\cdot 11$. Hence the ratio reduces to just $\mathbf{\frac{10}{11}}$. Note that an answer of $\frac{190}{209}$ should *not* receive credit.

2. There are currently 800 cells in the culture, and there will be 1600 cells six hours later. If the population grew steadily then there would be 1200 cells after three hours. Exponential growth, however, behaves differently. Recall that the graph of an exponential function through two points (such as e^x or 2^x) bends upward, always staying below the segment joining the same two points. Thus we expect the number of cells at the midpoint to be less than 1200, suggesting answer $\boldsymbol{A}$. (The precise answer is $800\cdot 2^{1/2}=800\sqrt{2}\approx 1131$, since $\sqrt{2}\approx 1.414$.)

3. The most convincing solution to this problem is to cut out nine small squares, draw an arrow on each, then set up the given arrow maze and follow the directions. After wandering around for a surprisingly long time within the little 3×3 grid, you should discover yourself retracing your initial path through the maze in reverse, finally exiting at square **7**. The final position of the arrows is shown at left, so that you can check yourself.



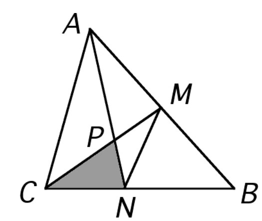
4. After distributing and rearranging, the given equation can be rewritten as $3x+bx=7b-21$. For most values of b, the only step remaining to solve the equation is to divide. For example, when $b=11$ we have $14x=56$, which gives $x=4$ upon dividing by 14. However, when $b=-3$ the x terms cancel, giving $0=-42$, which is not true for any value of x, so there is no solution when $b=\mathbf{-3}$.

5. We first demonstrate that there is a thrilling sequence of length six:

$$1,\ 2,\ -1,\ 28,\ -19,\ 100.$$

The sums of consecutive terms are 3, 1, 27, 9, and 81. To see that this is the shortest possible, suppose there were a three-term thrilling sequence of the form 1, a, 100. Then we would have $1+a=3^k$ and $a+100=3^l$ for integers $0\le k<l$. Subtracting these equations gives $99=3^l-3^k=3^k(3^{l-k}-1)$. Since 99 is divisible by 3^2 but no higher power of 3, we must have $k=2$, leading to $11=3^{l-k}-1$, or $3^{l-k}=12$, which doesn't work. Similar considerations rule out thrilling sequences of the form 1, a, b, 100; we would need $3^m-3^l+3^k=101$ in this case, which is not possible. (Proof?) Likewise a thrilling sequence of the form 1, a, b, c, 100 would imply $3^n-3^m+3^l-3^k=99$, which again cannot be done. Therefore the answer is indeed **6**.

6. Since M is the midpoint of $\overline{AB}$, we know that A is twice as high above $\overline{BC}$ as M, from which it follows that $area(ACN)=2area(MCN)$. However, there is a second method of comparing these two areas, using $\triangle PCN$, which is shaded in the diagram. We use the fact that the ratio of $area(ACN)$ to $area(PCN)$ is the same as the ratio of AN to PN. (Think of these segments as the bases of the two triangles to see why.) By the same token, the ratio of $area(PCN)$ to $area(MCN)$ is the same as the ratio of PC to MC. Letting $PM=x$, we have



$$\frac{area(ACN)}{area(MCN)}=\frac{area(ACN)}{area(PCN)}\cdot\frac{area(PCN)}{area(MCN)}=\frac{AN}{PN}\cdot\frac{PC}{MC}=\frac{10}{3}\cdot\frac{5}{x+5}.$$

We already know that this ratio equals 2, hence $6(x+5)=50$, which leads to $x=\mathbf{\frac{10}{3}}$. (*N.B.* There is also a short solution that employs the Theorem of Menelaus. The reader is invited to sort out the details.)

7. Suppose that there are s students and t groups in Mr. Strump's math class. Imagine listing the names of the three students in each group, a total of $3t$ names. But each student is in exactly two groups, so the list contains $2s$ names, implying that $3t = 2s$. Now consider the $\binom{t}{2} = \frac{1}{2}t(t-1)$ pairs of groups. According to the problem, one-third of these pairs involve groups with a student in common. It will be a different common student in each case, and every student will appear in this manner for some pair of groups. In other words, $\frac{1}{3}\binom{t}{2} = s$. Putting everything together, we have

$$\frac{1}{3}\binom{t}{2} = s = \frac{3t}{2} \qquad \Longrightarrow \qquad t(t-1) = 9t \qquad \Longrightarrow \qquad t = 10.$$

We conclude that there are **15** students in Mr. Strump's class.

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⋆ Regional Level ⋆

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PO Box 760
Potsdam, NY 13676

The Mandelbrot Competition
www.mandelbrot.org
info@mandelbrot.org